SI VIS PACEM, PARA BELLUM

Theory: § 1 and § 2.1 of DynamicModels

Two geopolitical blocks, West and East, are facing difficult times.
The authoritarian leaderships governing several countries of East are challenging West's role in international affairs, claiming that West takes advantage from its economical power in order to influence neutral countries and to put pressure on the other bloc.
On the other side, the democracies of West are responding that they inspire to the principles of freedom and self-determination, not obliging any other country to accept partnership and cooperation, but rather proposing them deals that are advantageous for both parties.
As a consequence of this situation, East has started to increase their military expenditures, hoping that showing strength would result in deterrence prompting West and the other countries to be more cautious while acting against the interests of East.
Being aware of this, West is considering to increase their military expenditures too, albeit it does not perceive any other action from East as hostile, apart from its rearm itself.
If the proposal of increment is approved, then the expenses for the future can be modeled as follows:

$$\begin{cases} x'(t) = y(t) - 2\,x(t) + 3 \\ y'(t) = x(t) - 2\,y(t) + 3 \end{cases}$$

with the initial conditions $x(0) = 1,\ y(0) = 2$.

Task 1: by performing the least possibile differential computations, what can be said about the evolution of this dynamic system in the short term?

Solution: by substituting, we obtain $x'(0) = 3,\ y'(0) = 0$. This means that West's expenditures will increase, while for inquiring East's ones we have to differentiate again: $y''(0) = x'(0) - 2\,y'(0) = 3$. The positivity of $y''(0)$ implies that $y'(t)$ is positive in a right neighbourhood of $t = 0$, giving evidence that also East's expenses will increase.

Task 2: does this system allow an equilibrium, i.e. a configuration that once reached would result in no more modifications of the situation over time?
If there is one, find it and provide comments about its meaning in our context.

Solution: by solving $x'(t) = y'(t) = 0$, we obtain $x(t) = y(t) = 3$. Note that the equilibrium refers to expenditures higher than those initially associated to the blocks.
Theory: the task can be repeated again after having shown to students § 2.2 of DynamicModels. Before that, they may try to solve the task differently from setting the derivatives as vanishing.

Task 3: is this equilibrium stable, in the sense that small perturbations result in returning to the original configuration, rather than moving away further? What does it entail, practically?

Solution: the product of the coefficients relative to the same function as in the LHS is $(-2)\cdot(-2)=4$, higher than the product involving swapped functions (that is $1\cdot 1=1$), so the equilibrium is stable.
As a consequence, if one or both block(s) alter the expenses due to external factors when at or near the equilibrium, according to the model the expenditures will tend to return as they were before the alteration.
Theory: the task can be repeated again after having shown to students § 2.3 of DynamicModels. Before that, they can try to conjecture stability by computing (x', y') for various (x, y) slightly different from $(3, 3)$.

Task 4: solve the system and devise how the blocks' expenditures will vary over time, by means of analytic considerations and geometric representations such as plots.

Solution: $x(t)=3-\frac{3}{2}\cdot\exp(-t)-\frac{1}{2}\cdot\exp(-3\,t)$, $y(t)=3-\frac{3}{2}\cdot\exp(-t)+\frac{1}{2}\cdot\exp(-3\,t)$. By differentiating, we obtain
$x'(t)=\frac{3}{2}\cdot\exp(-t)+\frac{3}{2}\cdot\exp(-3\,t)$, $y(t)=\frac{3}{2}\cdot\exp(-t)-\frac{3}{2}\cdot\exp(-3\,t)$.
Since $\exp(-t) > \exp(-3\,t) > 0$ for $t > 0$, both $x'(t)$ and $y'(t)$ are strictly positive for $t > 0$, so either blocks' expenditures will increase not only in the short term, generalizing what obtained in Task 1. The two expenses will tend asymptotically to 3, coming progressively closer to the equilibrium. This means the two blocks will increment their efforts, but only up to a certain extent.
Theory: the task can take advantage from § 2.4 of DynamicModels. Maple commands can be also used in order to ease the computational and graphical parts.

Task 5: Explore equilibria, stability and evolution of bivariate linear dynamical systems with the following system of interactive components.
Try to tune the coefficients, and give contextualized motivations for which it could happen.
Remember that a general bivariate linear dynamical system can be set up as follows.

$$\begin{cases} x'(t)=k y(t)-\alpha x(t)+p \\ y'(t)=l x(t)-\beta y(t)+q \end{cases}$$

$k=$ [1] , $\alpha=$ [2] , $p=$ [3]

$l=$ [1] , $\beta=$ [2] , $q=$ [3]

$x(0)=$ [1] , $y(0)=$ [2] , $t_{max}=$ [5]

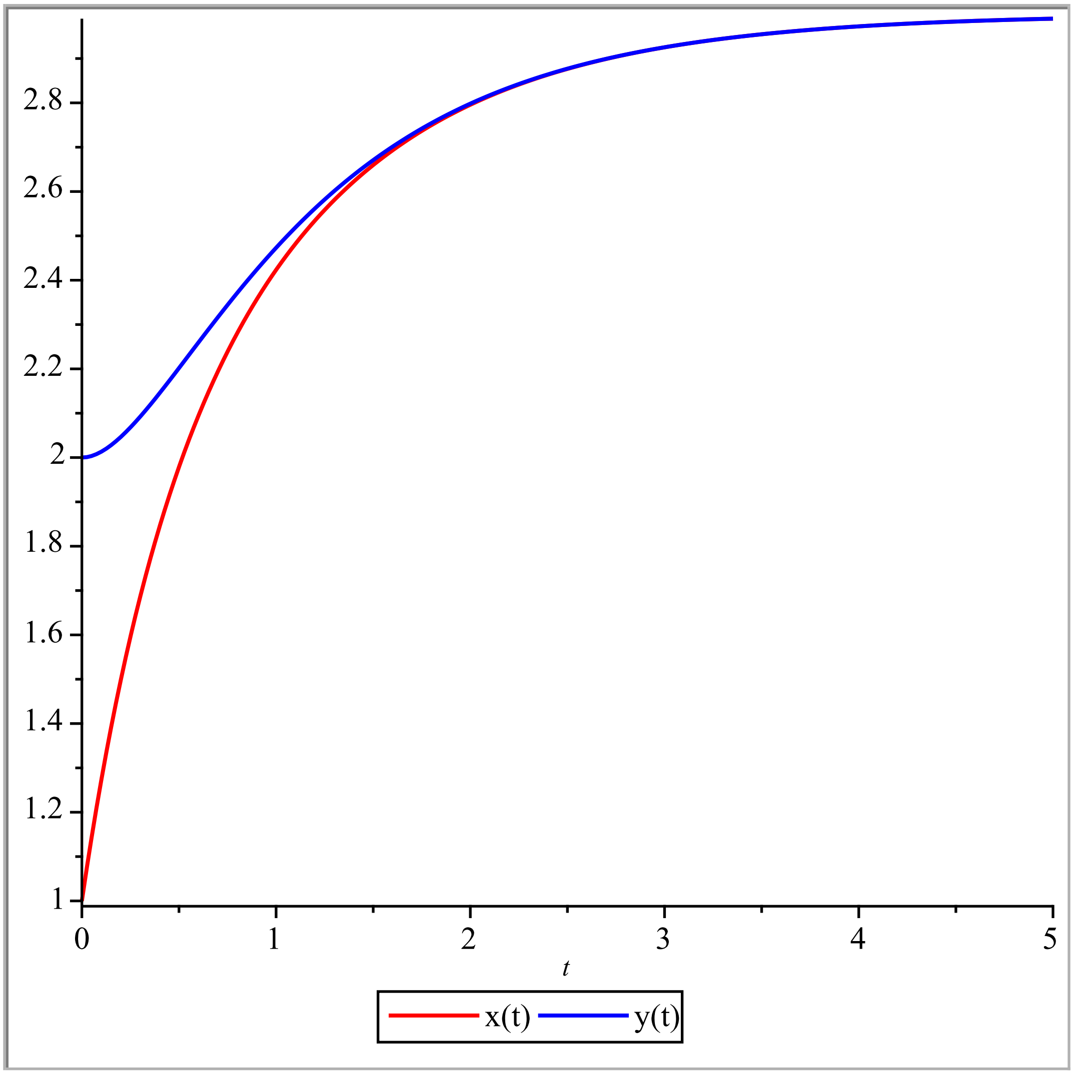
Compute!

Equilibrium: $x =$ 3 , $y =$ 3

Feasibility: yes

Stability: yes

Plot and analytic solution:



$$\left\{x(t) = -\frac{3\,e^{-t}}{2} - \frac{e^{-3t}}{2} + 3,\; y(t) = -\frac{3\,e^{-t}}{2} + \frac{e^{-3t}}{2} + 3\right\}$$