POST-WAR HEALTHCARE

After a war strucking it, the health situation of a country is needing attention.
In particular, an epidemic is spreading through the nation, eased by the marked difficulties in providing both prevention (e.g, vaccines) and medical support during wartime.
Since the end of the hostilities, relevant efforts have been carried out by the national authorities and also by some foreign organizations, in order to fight the disease.
To plan the interventions in the best possible way, it is pivotal to make predictions on how the epidemic will evolve over time, with the aim of assessing the impact of various ways of intervening.
This requires considering what the outcome would have been if no action had been taken, and the outcomes associated with taking action by different means.

Task 1: the most elementary mathematical model considers the number of new infections during a short timespan as proportional to the number of infected people.
Devise a differential equation representing this property, and solve it by finding proper functions satisfying this behaviour. Are these trends reliable over the long term? Why?

Solution: $y'(x) = k \cdot y(x)$, resulting in $y(x) = y(0) \cdot \exp(k \cdot x)$. While often reliable over the short term, it is not over the long term, since it exhibits unbounded growth, clashing with a finite population.

Task 2: in order to set up a better grounded model, it is useful to take into account that only people who have not yet been infected can be susceptible to infection.
Indeed, while the disease does not exclude at all the possibility of re-infections, they have been observed as being sufficiently rare that they can be neglected while developing this kind of models.
How it is possible to express these features in terms of differential equations? Provide some forms, by helping with numerical simulations if advantageous, and discuss merits and limitations.

Solution: the total number of infections cannot exceed the overall population, which can be denoted with M.
A first ODE can consist in, by starting from the equation as in Task 1, to set the increase proportional to the number of susceptible individuals.
This results in $y'(x) = k \cdot y(x) \cdot \left(1 - \frac{y(x)}{M}\right)$, corresponding to the logistic or Verhulst equation, and having solution $y(x) = \frac{M}{1 + q \cdot \exp(-k \cdot x)}$, where $q = \frac{M}{y(0)} - 1$.
Other approaches consider different functions as the last factor: for instance, the Gompertz equation takes the form $y'(x) = k \cdot y(x) \cdot \ln\left(\frac{M}{y(x)}\right)$, yielding $y(x) = M \cdot \exp\left(\ln\left(\frac{y(0)}{M}\right) \cdot \exp(-k \cdot x)\right)$.
Forms of this kind have the merit of having M as upper bound, thus respecting the population constraint.
However, some limitations remain, for instance because in a real epidemic usually not the whole population tend to be infected, and not every infected can infect other individuals for an indefinite amount of time.

Task 3: suppose that the nation has a population of 10 million, of which 6 can be infected (because the other 4 possess a natural immunity).
Assume also the epidemic expanding proportionally to both the number of infected and susceptible individuals.
How many people have been already infected when the highest number of daily infections will occur, that is when the healthcare system will be under the most pressure?
Discuss this issue both graphically and analitically, by observing that it would be important to intervene prior to that time, in order to ensure that the healthcare system does not become overburdened.

Solution: by using the Verhulst equation with $M = 6$, we obtain the solution
$y(x) = \dfrac{6}{1 + q \cdot \exp(-k \cdot x)}$; its first two derivatives are:
$y'(x) = \dfrac{6 \cdot k \cdot q \cdot \exp(k \cdot x)}{(\exp(k \cdot x) + q)^2}$ and $y''(x) = \dfrac{6 \cdot \exp(k \cdot x) \cdot k^2 \cdot q \cdot (q - \exp(k \cdot x))}{(\exp(k \cdot x) + q)^3}$, with the latter vanishing for x such that $\exp(k \cdot x) = q$; let us call it x^*.

By substituting, we have $y(x^*) = \dfrac{6}{1 + q \cdot \dfrac{1}{q}} = \dfrac{6}{2} = 3$, so the inflection point occurs when the infected are 3 million, namely half of the initially susceptible population.
It could be easily seen that this point is the maximum of $y'(x)$, thus representing when the daily infections are the highest.
Note how the fact $y''(x^*) = 0 \Rightarrow y(x^*) = 3$ (and equal to $\dfrac{M}{2}$ for generic M) is independent from the intially infected $y(0)$ and the parameter k, albeit the values of x^* and $y'(x^*)$ do actually depend from $y(0)$ and k.

Task 4: having determined that, according to the predictions, the healthcare system would dangerously suffer while dealing with the epidemic peak, the authorities and the organizations are planning interventions on two fronts. While continuing to give medical support to the infected requiring it, on the one hand they are going to implement social distancing, that is closing crowded venues such as dance halls and stadia, limiting access to restaurants and bars, and encouraging remote working. On the other hand, they are speeding up the distribution and the administering of vaccines, in order to give progressively immunity to more people.
By considering the previous models, how these actions will impact on the epidemic evolution? Try to answer this question both theoretically and with practical examples.

Solution: the social distancing slowens the epidemic spread, not preventing healthy individuals to get infected, but ensuring that the daily number of new infections is lower than the one resulting from not implementing that measure, thus implying less pressure on the healthcare system. This can be modeled with a reduction in the value of k.

The faster administering of vaccines reduces the susceptible population regardless of the epidemic dynamics, thus being representable by reducing the value of M.

Task 5: the following systems of interactive components allows for generalization:

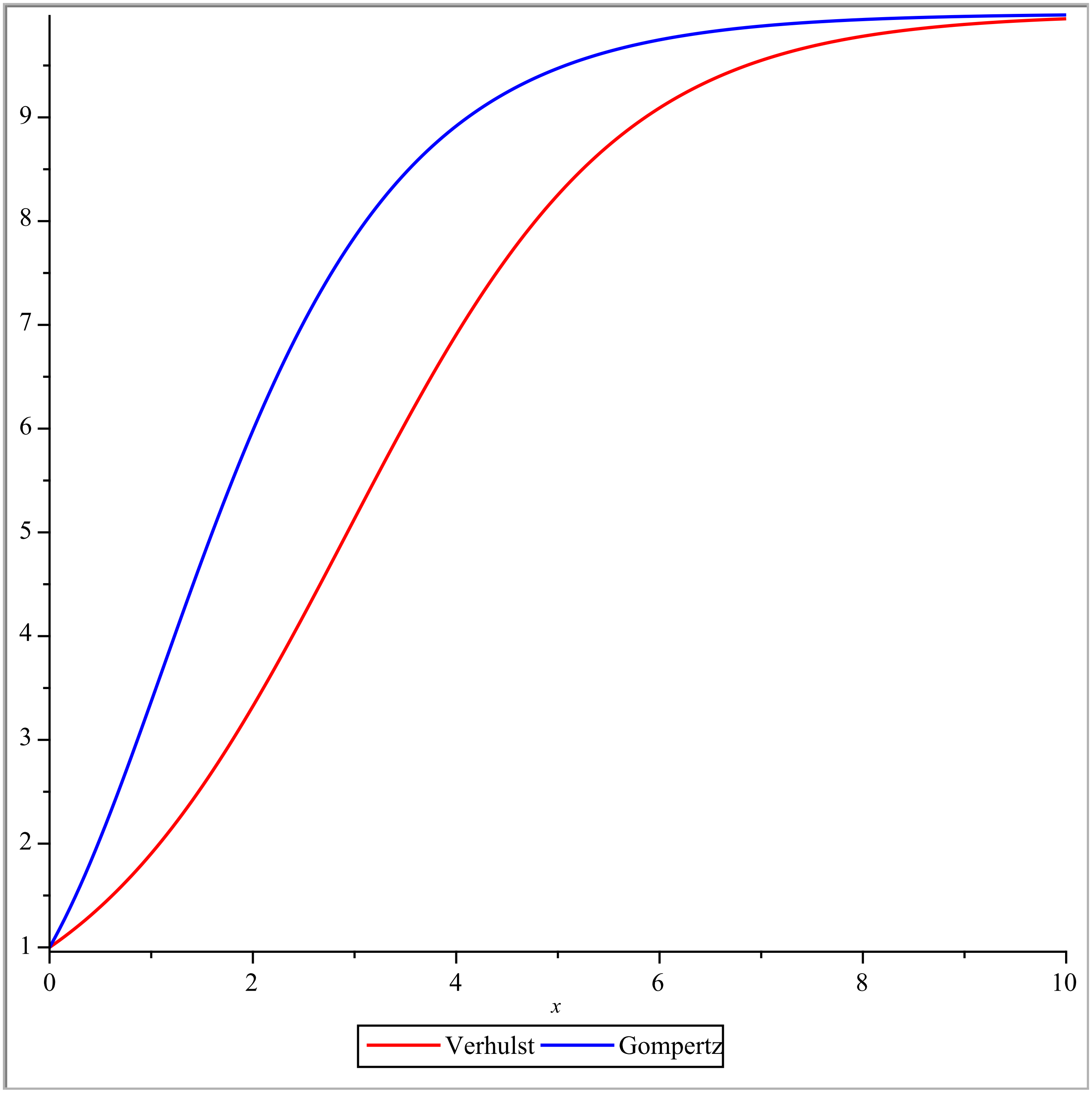
Initially infected 1

Initially susceptible population 10

Expansion parameter 0.75

Maximum time 10

Compute!



Try to vary the quantities according to proper contexts, and comment the results.